A Proof of Cantor-Bernstein-Schröder

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February 6, 2017

We present a proof of Cantor-Bernstein-Schröder based on Knaster's argument in [1]. The proof is given at a level of detail sufficient to prepare the reader to consider corresponding formal proofs in interactive theorem provers.

Definition 1. *Let $\Phi : \wp(A) \to \wp(B)$. We say Φ is* monotone *if $\Phi(U) \subseteq \Phi(V)$ forall $U, V \in \wp(A)$ such that $U \subseteq V$. We say Φ is* antimonotone *if $\Phi(V) \subseteq \Phi(U)$ forall $U, V \in \wp(A)$ such that $U \subseteq V$.*

Definition 2. *For sets A and B we write $A \setminus B$ for $\{u \in A | u \notin B\}$.*

Lemma 3. *Let A be a set and $\Phi : \wp(A) \to \wp(A)$ be given by $\Phi(X) = A \setminus X$. Then Φ is antimonotone.*

Proof. Left to reader. □

Definition 4. *Let $f : A \to B$ be a function from a set A to a set B. For $X \in \wp(A)$ we write $f(X)$ for $\{f(x)|x \in A\}$.*

Lemma 5. *Let $f : A \to B$ be a function from a set A to a set B. Let $\Phi : \wp(A) \to \wp(B)$ be given by $\Phi(X) = f(X)$. Then Φ is monotone.*

Proof. Left to reader. □

Theorem 6 (Knaster-Tarski Fixed Point)**.** *Let $\Phi : \wp(A) \to \wp(A)$. Assume Φ is monotone. Then there is some $Y \in \wp(A)$ such that $\Phi(Y) = Y$.*

Proof. Let Y be $\{u \in A | \forall X \in \wp(A).\Phi(X) \subseteq X \to u \in X\}$. The following is easy to see:

$$Y \subseteq X \text{ for all } X \in \wp(A) \text{ such that } \Phi(X) \subseteq X. \tag{1}$$

We prove $\Phi(Y) \subseteq Y$ and $Y \subseteq \Phi(Y)$.

We first prove $\Phi(Y) \subseteq Y$. Let $u \in \Phi(Y)$. We must prove $u \in Y$. Let $X \in \wp(A)$ such that $\Phi(X) \subseteq X$ be given. By (1) $Y \subseteq X$. Hence $\Phi(Y) \subseteq \Phi(X)$ by monotonicity of Φ. Since $u \in \Phi(Y)$, we have $u \in \Phi(X)$. Since $\Phi(X) \subseteq X$, we conclude $u \in X$.

We next turn to $Y \subseteq \Phi(Y)$. Since $\Phi(Y) \subseteq Y$, we know $\Phi(\Phi(Y)) \subseteq \Phi(Y)$ by monotonicity of Φ. Hence $Y \subseteq \Phi(Y)$ by (1). □

Definition 7. *Let $f : A \to B$ be a function. We say f is* injective *if $\forall xy \in A.f(x) = f(y) \to x = y$.*

Definition 8. *We say sets A and B are* equipotent *if there exists a relation R such that*

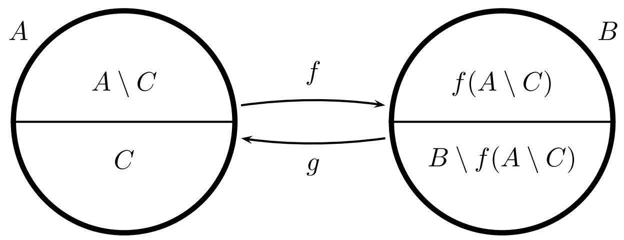
1. $\forall x \in A.\exists y \in B.(x, y) \in R$
2. $\forall y \in B.\exists x \in A.(x, y) \in R$
3. $\forall x \in A.\forall y \in B.\forall z \in A.\forall w \in B.(x, y) \in R \wedge (z, w) \in R \rightarrow (x = z \iff y = w)$

Theorem 9 (Cantor-Bernstein-Schröder)**.** *If $f : A \rightarrow B$ and $g : B \rightarrow A$ are injective, then A and B are equipotent.*

Proof. Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be given injective functions. Let $\Phi : \wp(A) \rightarrow \wp(A)$ be defined by $\Phi(X) = g(B \setminus f(A \setminus X))$. It is easy to see that Φ is monotone by Lemmas 3 and 5. By Theorem 6 there is some $C \in \wp(A)$ such that $\Phi(C) = C$. Hence $C \subseteq A$ and

$$\forall x.x \in C \iff x \in g(B \setminus f(A \setminus C)). \tag{2}$$

We can visualize the given information as follows:



Let $R = \{(x, y) \in A \times B | x \notin C \wedge y = f(x) \vee x \in C \wedge x = g(y)\}$. We must prove the three conditions in Definition 8.

1. Let $x \in A$ be given. We must find some $y \in B$ such that $(x, y) \in R$. We consider cases based on whether $x \in C$ or $x \notin C$. If $x \notin C$, then we can take y to be $f(x)$. Assume $x \in C$. By (2) we know $x \in g(B \setminus f(A \setminus C))$. Hence there is some $y \in B \setminus f(A \setminus C)$ such that $x = g(y)$ and we can use this y as the witness.

2. Let $y \in B$ be given. We must find some $x \in A$ such that $(x, y) \in R$. We consider cases based on whether or not $y \in f(A \setminus C)$. If $y \in f(A \setminus C)$, then there is some $x \in A \setminus C$ such that $f(x) = y$ and we can use this same x as the witness. Assume $y \notin f(A \setminus C)$. Note that $g(y) \in C$ using 2 and $y \in B \setminus f(A \setminus C)$. Hence we can take $g(y)$ as the witness.

3. Before proving the third property, we prove the following claim:

$$\forall x \in A.\forall y \in B.x \in C \wedge x = g(y) \rightarrow y \notin f(A \setminus C) \tag{3}$$

Let $x \in A$ and $y \in B$ be given. Assume $x \in C$, $x = g(y)$ and $y \in f(A \setminus C)$. Since $x \in C$,there is some $w \in B \setminus f(A \setminus C)$ such that $g(w) = x$ by (2). Since g is injective, $w = y$ contradicting $y \in f(A \setminus C)$.

Now that we know (3) we can easily prove the third property by splitting into four cases. Let $x \in A$, $y \in B$, $z \in A$ and $w \in B$ be given. Assume $(x, y) \in R$ and $(z, w) \in R$. By the definition of R there are two cases for $(x, y) \in R$ and two cases for $(z, w) \in R$. In each case we need to prove $x = z \iff y = w$.

- Assume $x \notin C$, $y = f(x)$, $z \notin C$ and $w = f(z)$. The fact that $x = z \iff y = w$ follows easily from injectivity of f.
- Assume $x \notin C$, $y = f(x)$, $z \in C$ and $z = g(w)$. In order to prove $x = z \iff y = w$ we argue that $x \neq z$ and $y \neq w$. Clearly $x \neq z$ since $x \notin C$ and $z \in C$. By (3) we know $w \notin f(A \setminus C)$. On the other hand $y \in f(A \setminus C)$ since $y = f(x)$ and $x \in A \setminus C$. Hence $y \neq w$.
- Assume $x \in C$, $x = g(y)$, $z \notin C$ and $w = f(z)$. Again in order to prove $x = z \iff y = w$ we argue that $x \neq z$ and $y \neq w$. Clearly $x \neq z$ since $x \in C$ and $z \notin C$. By (3) we know $y \notin f(A \setminus C)$. On the other hand $w \in f(A \setminus C)$ since $w = f(z)$ and $z \in A \setminus C$. Hence $y \neq w$.
- Assume $x \in C$, $x = g(y)$, $z \in C$ and $z = g(w)$. The fact that $x = z \iff y = w$ follows easily from injectivity of g.

□

Corollary 10. *If $f : A \to B$ is injective and $B \subseteq A$, then A and B are equipotent.*

Proof. This follows immediately from Theorem 9 using the injection from B into A, since this injection is obviously injective. □

References

[1] B. Knaster. Un théorème sur les fonctions d'ensembles. *Ann. Soc. Polon. Math*, 6:133–134, 1928.