Towards Machine Learning for SMT

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Outline

Intro: QBF, Expansion, Games, Careful expansion

Solving QBF

Learning in QBF

Towards SMT

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SAT and QBF

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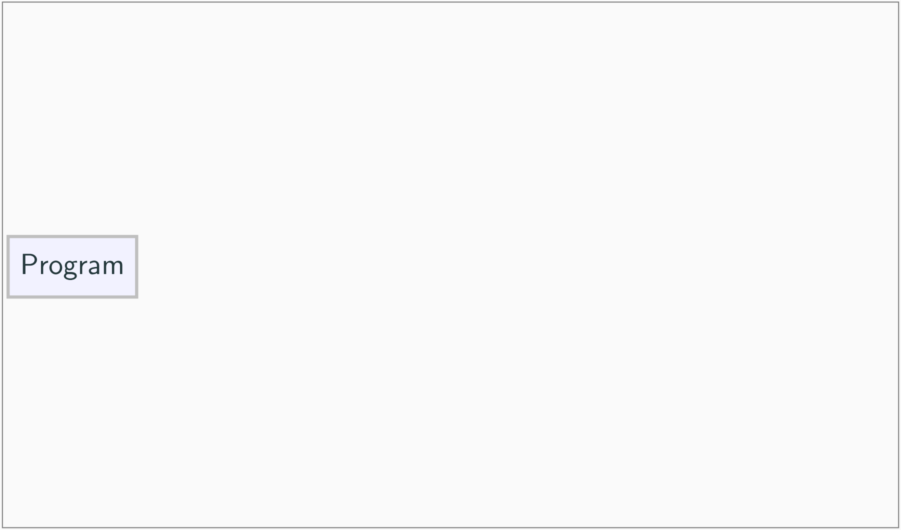
- focus: software verification, debugging, ...
- formulas may be huge, solvers are not based on saturation but on SAT technology

How is SMT Used in SW Verification



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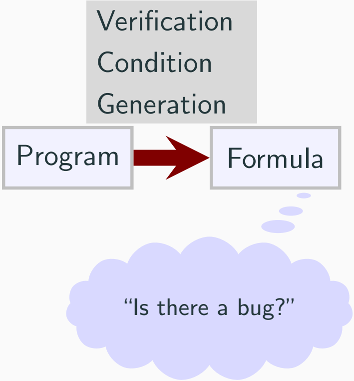

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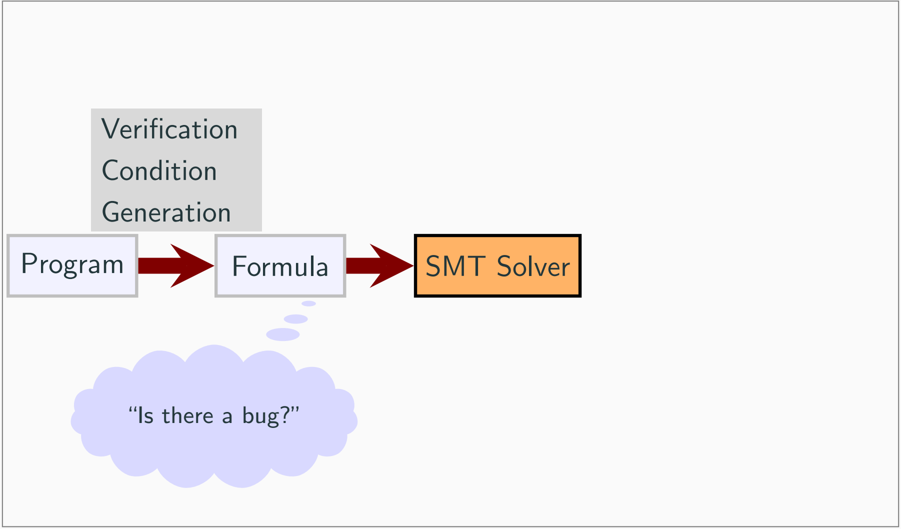

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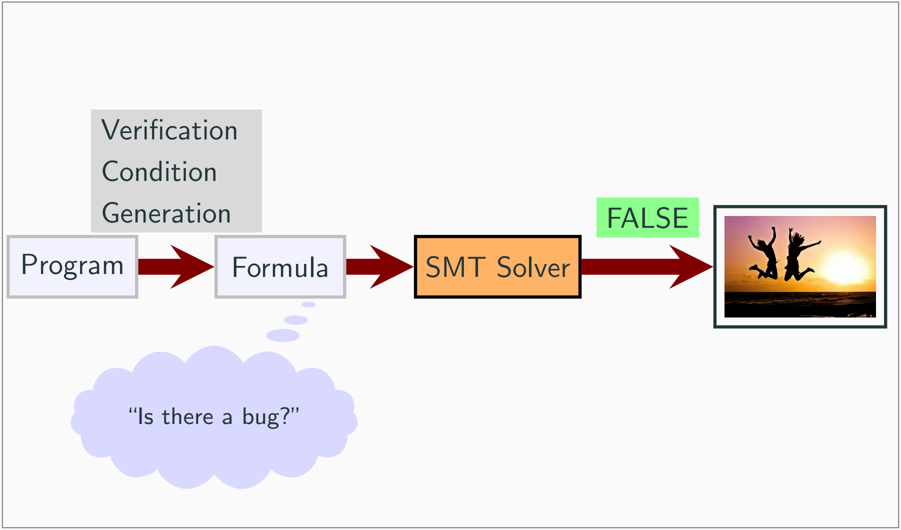

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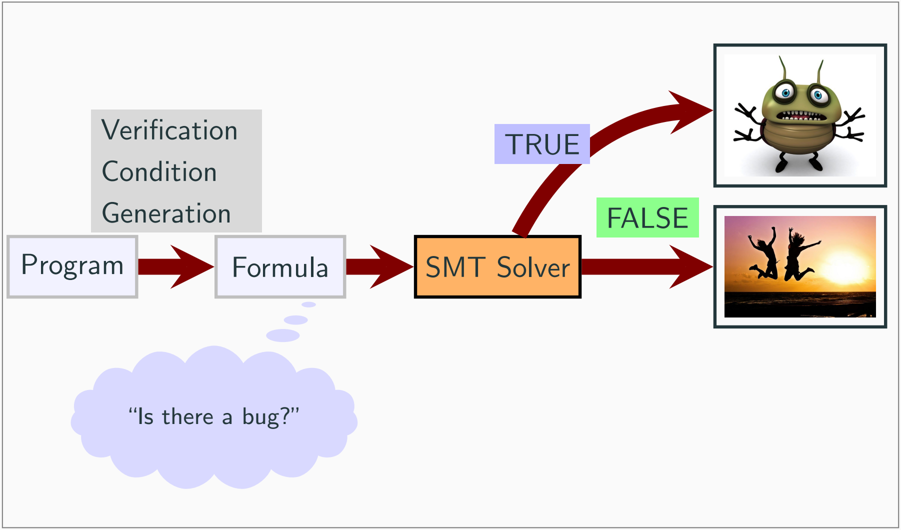

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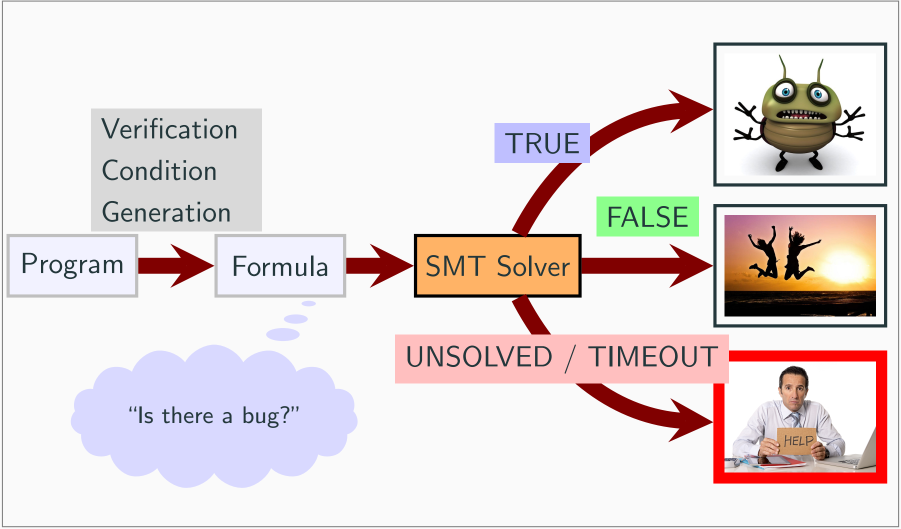

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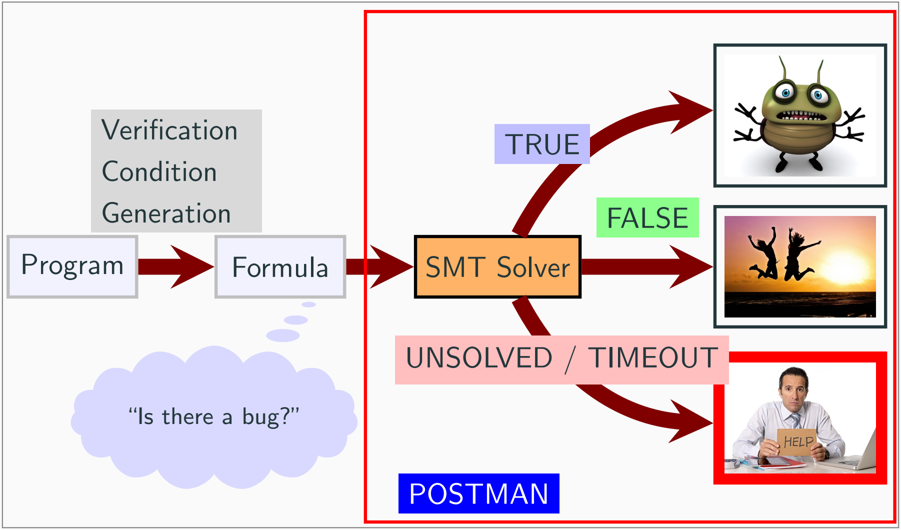
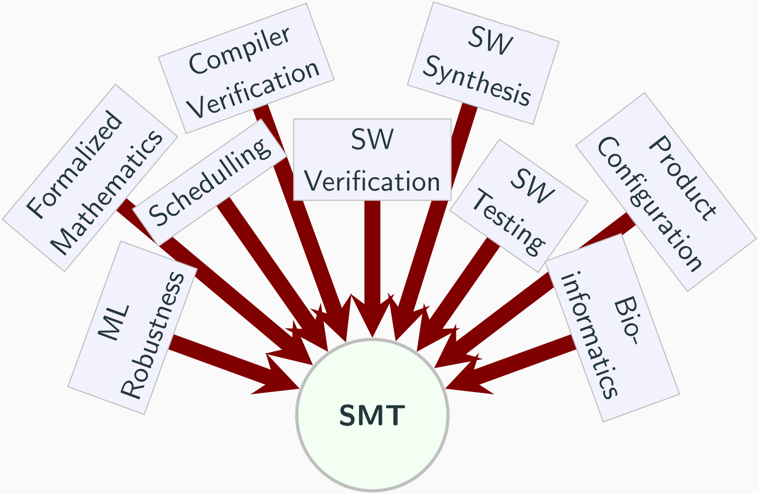


SMT Impact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Compiler Verification
Schedulling
SW Verification
SW Synthesis
SW Testing
Product Configuration
ML Robustness
Bio-informatics
SMT

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Hiring postdocs and PhD students to work on SMT + ML.

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$$(\forall X_u.\ \text{is-true}(X_u) \leftrightarrow p_e(X_u))$$

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- Alternatively, use equality:

$$t \neq f \wedge (\forall X_u.\ (X_u = t) \leftrightarrow p_e(X_u))$$

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  Example

$$\forall u \exists e. (u \leftrightarrow e)$$

$\exists$-player wins by playing $e \triangleq u$.

Solving QBF

Solving by CEGAR Expansion

$$\exists \mathcal{E}\, \forall \mathcal{U}.\ \phi \equiv \exists \mathcal{E}.\ \bigwedge_{\mu \in 2^{\mathcal{U}}} \phi[\mu]$$

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Observe:

$$\exists \mathcal{E}.\ \bigwedge_{\mu \in 2^{\mathcal{U}}} \phi[\mu] \Rightarrow \exists \mathcal{E}.\ \bigwedge_{\mu \in \omega} \phi[\mu]$$

for some $\omega \subseteq 2^{\mathcal{U}}$

What is a good ω?

Solving by CEGAR Expansion Contd.

$$\exists \mathcal{E}\, \forall \mathcal{U}.\ \phi \equiv \exists \mathcal{E}.\ \bigwedge_{\mu \in 2^{\mathcal{U}}} \phi[\mu]$$

Expand gradually instead: [J. and Marques-Silva, 2011]

- Pick τ_0 arbitrary assignment to $\mathcal{E}$

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- $\mathtt{SAT}(\phi[\mu_0] \wedge \phi[\mu_1]) = \tau_2$ assignment to $\mathcal{E}$
- After n iterations

$$\exists \mathcal{E}.\ \bigwedge_{i \in 1..n} \phi[\tau_i]$$

Abstraction-Based Algorithm for a Winning Move

Algorithm for $\exists\forall$. Generalize to arbitrary number of alternations using recursion. [J. et al., 2012].

Function $\texttt{Solve}(\exists X \forall Y.\, \phi)$

$\alpha \leftarrow \text{true}$ // start with an empty abstraction

while true **do**

  $\tau \leftarrow \texttt{SAT}(\alpha)$ // find a candidate

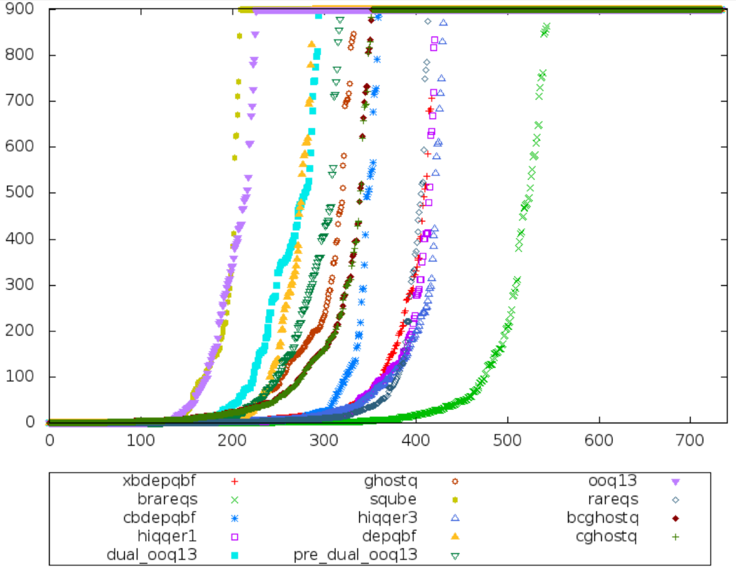
  if $\tau = \bot$ **then return** $\bot$

  $\mu \leftarrow \texttt{Solve}(\neg\phi[X \leftarrow \tau])$ // find a countermove

  if $\mu = \bot$ **then return** τ

  $\alpha \leftarrow \alpha \wedge \phi[Y \leftarrow \mu]$ // refine abstraction

Results, QBF-Gallery '14, Application Track



Careful Expansion: Good Example

$$\exists x \ldots \forall y \ldots . \; \phi \wedge y$$

Setting countermove $y \leftarrow 0$ yields false. Stop.

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$$\exists x \ldots \forall y \ldots . \; x \vee \phi$$

Setting candidate $x \leftarrow 1$ yields true (impossible to falsify). Stop.

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4. $\mathtt{SAT}(\neg(0 \Leftrightarrow y)) \ldots y \leftarrow 1$ countermove
5. $\mathtt{SAT}(x \Leftrightarrow 0 \wedge x \Leftrightarrow 1) \ldots$ UNSAT Stop

Careful Expansion: Ugly Example

$$\exists x_1 x_2 \forall y_1 y_2.\ x_1 \Leftrightarrow y_1 \vee x_2 \Leftrightarrow y_2$$

1. $x_1, x_2 \leftarrow 0, 0$

Careful Expansion: Ugly Example

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6. $\ldots$

Learning in QBF

Issue

- CEGAR requires 2^n SAT calls for the formula

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$$\left(\exists x_1 \dots x_n \forall y_1 \dots y_n. \bigvee_{i \in 1..n} x_i \Leftrightarrow \neg x_i\right) \equiv \left(\exists x_1 \dots x_n. \; 0\right)$$

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- Idea: instead of plugging in constants, plug in functions.
- **Where do we get the functions?**

Use Machine Learning

[J., 2018]

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4. Repeat.

Machine Learning Example

x_1	x_2	$\dots$	x_n	y_1	y_2	$\dots$	y_n
0	0	$\dots$	0	1	1	$\dots$	1
1	0	$\dots$	0	0	1	$\dots$	1
0	0	$\dots$	1	1	1	$\dots$	0
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- After 2 steps: $y_1 \leftarrow \neg x_1$, $y_i \leftarrow 1$ for $i \in 2..n$.

Machine Learning Example

x_1	x_2	...	x_n	y_1	y_2	...	y_n
0	0	...	0	1	1	...	1
1	0	...	0	0	1	...	1
0	0	...	1	1	1	...	0
0	1	...	1	1	0	...	0

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- Eventually we learn the right functions.

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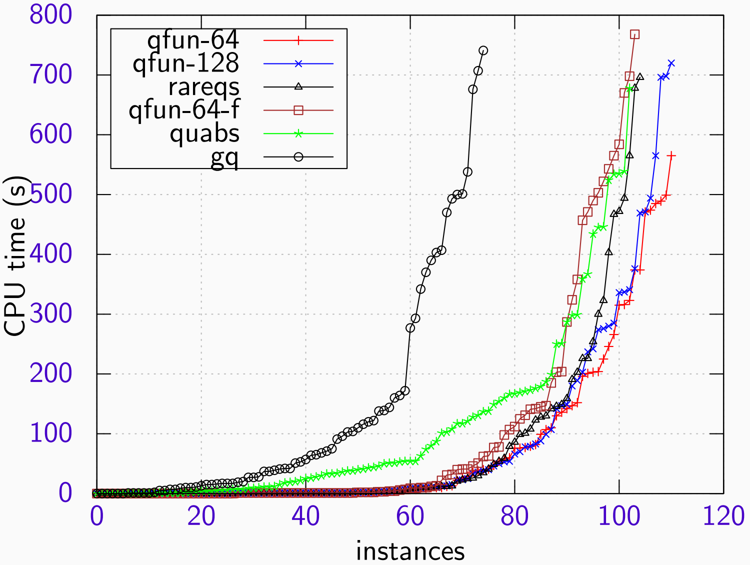
- Additional heuristic: If a learned function still works, keep it. "Don't fix what ain't broke."

Current Implementation: Experiments



Towards SMT

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- in SMT we always have equality
- almost always need uninterpreted functions
- Challenge: learning uninterpreted functions
- looking at finite models

$$(\forall x)(\mathsf{range}(x) \rightarrow \mathsf{memory}(x) = c)$$

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- Therefore we can look for a model with the universe $*_1, \ldots, *_{n'}$, $n' \leq n$.

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p_i predicates, c_i constants, X variables

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7. GOTO 2

Learning in Finite Models' CEGAR

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4. *Learn* entire interpretation from observing values of existing terms.

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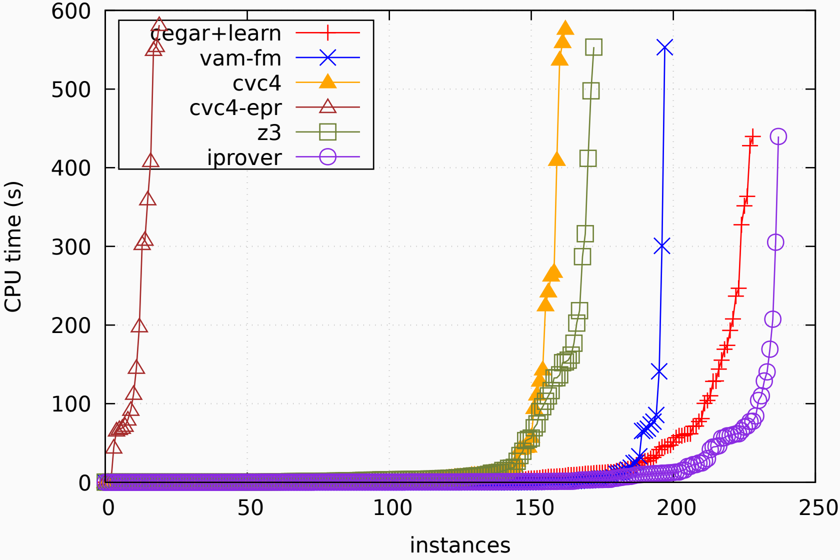
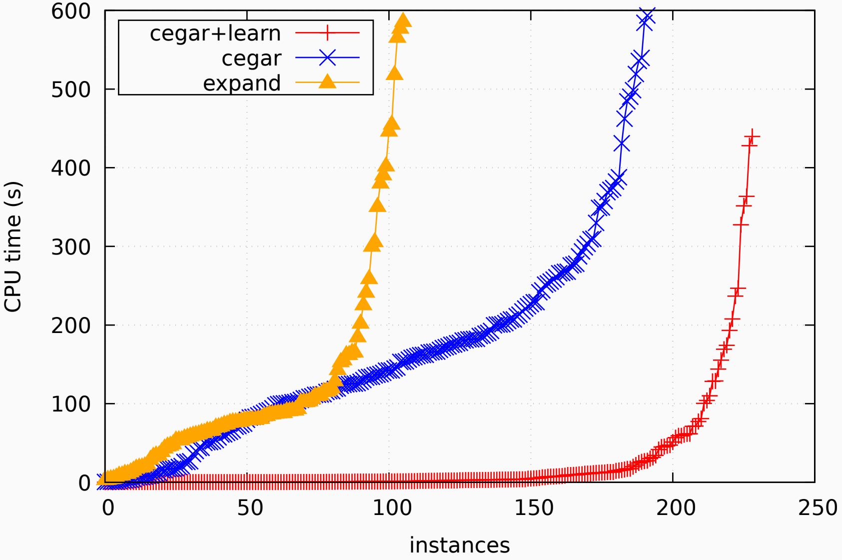
5. Learn:
   $t \triangleq *_1$
   $p(X_1, \ldots, X_n) \triangleq (X_1 = *_1)$

Results EPR

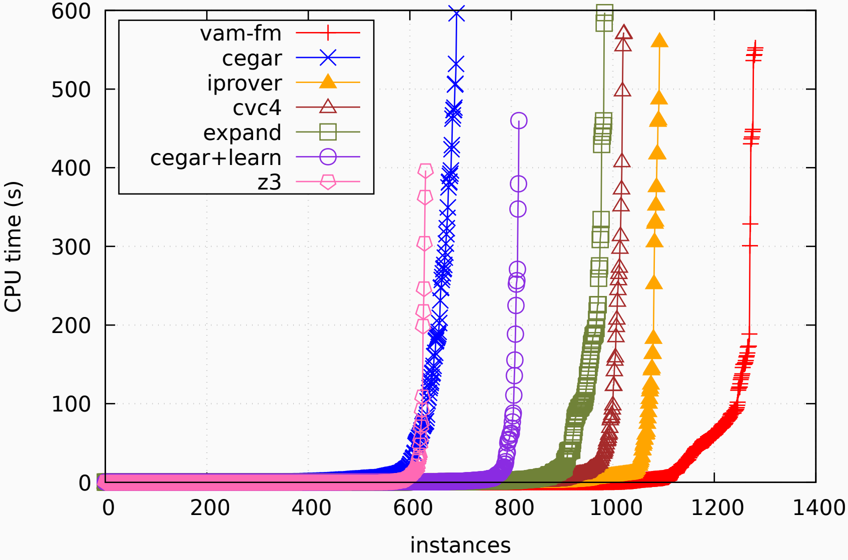


600
500
400
300
200
100
0
CPU time (s)
0
50
100
150
200
250
instances
cegar+learn
vam-fm
cvc4
cvc4-epr
z3
iprover

Results EPR: QFM



Results SAT NON-EPR



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Thank You for Your Attention!

Questions?

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