Saturation-based Theorem Proving and ML

Course Machine Learning and Reasoning 2020

MLR 2020[1]

[1]Czech Technical Univeristy in Prague (CIIRC)

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Outline

Arbitrary First-Order Formulas

- A first-order signature (vocabulary): function symbols (including constants), predicate symbols. Equality is part of the language.
- A set of variables.
- Terms are built using variables and function symbols. For example, $f(x) + g(x)$.
- Atoms, or atomic formulas are obtained by applying a predicate symbol to a sequence of terms. For example, $p(a, x)$ or $f(x) + g(x) \geq 2$.
- Formulas: built from atoms using logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and quantifiers $\forall$, $\exists$. For example, $(\forall x)x = 0 \vee (\exists y)y > x$.

Clauses

- Literal: either an atom A or its negation $\neg A$.
- Clause: a disjunction $L_1 \vee \ldots \vee L_n$ of literals, where $n \geq 0$.

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- Empty clause, denoted by $\square$: clause with 0 literals, that is, when $n = 0$.
- A formula in Clausal Normal Form (CNF): a conjunction of clauses.
- A clause is ground if it contains no variables.
- If a clause contains variables, we assume that it implicitly universally quantified. That is, we treat $p(x) \vee q(x)$ as $\forall x(p(x) \vee q(x))$.

What an Automatic Theorem Prover is Expected to Do

Input:

- a set of axioms (first order formulas) or clauses $\mathcal{A}$
- a conjecture (first-order formula or set of clauses) G

Question:

- Does G logically follow from $\mathcal{A}$?

$$\mathcal{A} \overset{?}{\models} G$$

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Output:

- Either yes and a proof,
- or . . . ?

Proof by Refutation

Given a problem with axioms and assumptions $\mathcal{A} = F_1, \ldots, F_n$ and conjecture G,

1. negate the conjecture;
2. establish unsatisfiability of the set of formulas $F_1, \ldots, F_n, \neg G$.

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Thus, we reduce the theorem proving problem to the problem of checking unsatisfiability.

General Scheme in One Slide

- Read a problem P
- Preprocess the problem: $P \Longrightarrow P'$
- Convert P' into Clause Normal Form N
  - replacing connectives, formula naming, distributive laws
  - Skolemisation
- Run a saturation algorithm on it, try to derive $\square$.
  - computes a closure of N with respect to an inference system
  - logical calculus: resolution + superposition
- If $\square$ is derived, report the result, maybe including a refutation.

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Trying to derive $\square$ using a saturation algorithm is the hardest part, which in practice may not terminate or run out of memory.

A Bit More on the CNF Transformation

- replacing unwanted connectives:

$$
\begin{array}{lcl}
A \leftrightarrow B & \Longrightarrow & (A \rightarrow B) \wedge (B \rightarrow A) \\
A \rightarrow B & \Longrightarrow & \neg A \vee B \\
\neg(A \vee B) & \Longrightarrow & \neg A \wedge \neg B \\
 & \ldots &
\end{array}
$$

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- distributive laws:

$$(A \wedge B) \vee (C \wedge D) \Longrightarrow (A \vee C) \wedge (A \vee D) \wedge (B \vee C) \wedge (B \vee D)$$

A Bit More on the CNF Transformation

- replacing unwanted connectives:

$$\begin{array}{lcl} A \leftrightarrow B & \Longrightarrow & (A \rightarrow B) \land (B \rightarrow A) \\ A \rightarrow B & \Longrightarrow & \neg A \lor B \\ \neg(A \lor B) & \Longrightarrow & \neg A \land \neg B \\ & \cdots & \end{array}$$

- distributive laws:

$$(A \land B) \lor (C \land D) \Longrightarrow (A \lor C) \land (A \lor D) \land (B \lor C) \land (B \lor D)$$

- formula naming (Tseitin / Pleisted-Greenbaum):

$$(A \land B) \lor (C \land D) \Longrightarrow (F_{AB} \lor (C \land D)) \land (F_{AB} \rightarrow A) \land (F_{AB} \rightarrow B)$$

A Bit More on the CNF Transformation

- replacing unwanted connectives:

$$\begin{array}{lcl} A \leftrightarrow B & \Longrightarrow & (A \to B) \land (B \to A) \\ A \to B & \Longrightarrow & \neg A \lor B \\ \neg(A \lor B) & \Longrightarrow & \neg A \land \neg B \\ & \cdots & \end{array}$$

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- Skolemisation on an example

$$\forall x[x \neq 0 \to \exists y(x \cdot y = 1)] \quad \Longrightarrow \quad x \neq 0 \to x \cdot sk_y(x) = 1$$

The Premise Selection Task

The set of clauses $F_1, \ldots, F_n, \neg G$ to be passed to the saturation may be too large to process efficiently

- common sense reasoning tasks (big ontologies)
- automatic support for interactive provers
  - e.g. Mizar, Isabelle, HOL, and Coq
  - large background libraries of already formalized math

Premise Selection:

- heuristically pick a subset $\mathcal{A}' \subset \mathcal{A} = F_1, \ldots, F_n$ such that $\mathcal{A}', \neg G$ is (likely) still unsatisfiable

Approaches to Premise Selection

"Traditional" – SInE:

- The SUMO Inference Engine
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- Premise Selection for Mathematics by Corpus Analysis and Kernel Methods. J. Autom. Reasoning (2014)
- DeepMath - Deep Sequence Models for Premise Selection. NIPS 2016
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Learning from previously discovered proofs

Outline

1. First-order Logic and Theorem Proving
2. Saturation-based Proving
3. Further Tuning and the Role of Strategies
4. Summary

Overview

Saturation-based proving

- the most prominent technology for proving in FOL
  - provers: E, Vampire, Spass, iProver, ...
- alternatives:
  - the tableaux approach: e.g. LeanCop
  - Satisfiability Modul Theories (SMT): Z3, CVC4, . . .

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Topics:

- A Static View: Inferences, Soundness, and Completeness
- A Dynamic View: The Saturation Loop
- Making It Fast in Practice

Inference System

- An inference has the form

$$\frac{F_1 \quad \dots \quad F_n}{G},$$

  where $n \geq 0$ and $F_1, \dots, F_n, G$ are formulas (clauses).
- The formula G is called the conclusion of the inference;
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- An inference rule R is a set of inferences.
- Every inference $I \in R$ is called an instance of R.
- An Inference system / calculus $\mathbb{I}$ is a set of inference rules.

Derivation, Proof

- Derivation in an inference system $\mathbb{I}$:
  a DAG built from inferences in $\mathbb{I}$.
- Derivation of E from $E_1, \ldots, E_m$: a finite derivation of E whose every leaf is one of the expressions $E_1, \ldots, E_m$ and the root of which is is E.
- A refutation is a derivation of the empty clause $\square$.

The superposition calculus

Resolution **Factoring**

$$\frac{\underline{A} \vee C_1 \quad \underline{\neg A'} \vee C_2}{(C_1 \vee C_2)\theta}, \qquad \frac{\underline{A} \vee A' \vee C}{(A \vee C)\theta},$$

where, for both inferences, $\theta = \mathsf{mgu}(A, A')$ *and* A *is not an equality literal*

Superposition

$$\frac{\underline{l \simeq r} \vee C_1 \quad \underline{L[s]_p} \vee C_2}{(L[r]_p \vee C_1 \vee C_2)\theta} \quad \text{or} \quad \frac{\underline{l \simeq r} \vee C_1 \quad \underline{t[s]_p \otimes t'} \vee C_2}{(t[r]_p \otimes t' \vee C_1 \vee C_2)\theta},$$

where $\theta = \mathsf{mgu}(l, s)$ *and* $r\theta \not\succeq l\theta$ *and, for the left rule* $L[s]$ *is not an equality literal, and for the right rule* $\otimes$ *stands either for* $\simeq$ *or* $\not\simeq$ *and* $t'\theta \not\succeq t[s]\theta$

EqualityResolution **EqualityFactoring**

$$\frac{\underline{s \not\simeq t} \vee C}{C\theta}, \qquad \frac{\underline{s \simeq t} \vee s' \simeq t' \vee C}{(t \not\simeq t' \vee s' \simeq t' \vee C)\theta},$$

where $\theta = \mathsf{mgu}(s, t)$

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Fig. 1. The rules of the superposition and resolution calculus.

Soundness and Completeness

Soundness

- An inference is sound if the conclusion of this inference is a logical consequence of its premises.
- An inference system is sound if every inference rule in this system is sound.

Consequence of soundness: Let S be a set of clauses. If $\square$ can be derived from S by a sound $\mathbb{I}$ then S is unsatisfiable.

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Completeness

An inference system $\mathbb{I}$ is complete, if for every unsatisfiable set of clauses S, there is a derivation of $\square$ from S using $\mathbb{I}$.

Idea of Saturation

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Idea:

- Take a set of clauses S (the *search space*), initially $S = S_0$. Repeatedly apply inferences in $\mathbb{I}$ to clauses in S and add their conclusions to S, unless these conclusions are already in S.
- If, at any stage, we obtain $\square$, we terminate and report unsatisfiability of S_0.

Saturation Algorithm

A saturation algorithm tries to *saturate* a set of clauses with respect to a given inference system.

In theory there are three possible scenarios:

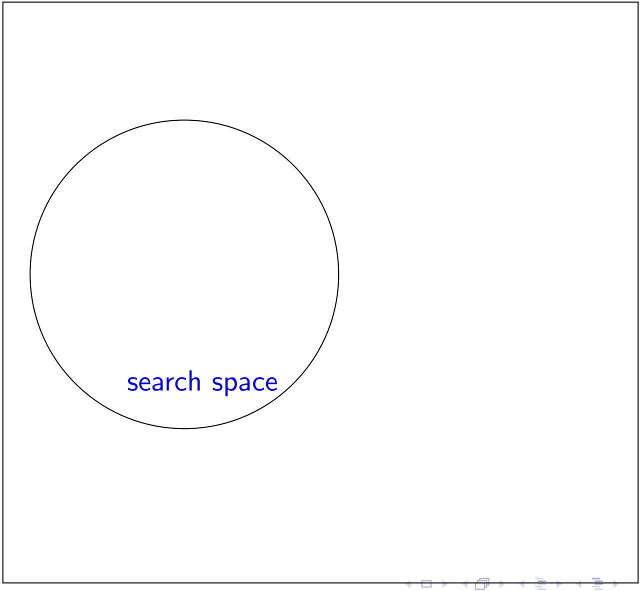
1. At some moment the empty clause $\square$ is generated, in this case the input set of clauses is unsatisfiable.
2. Saturation will terminate without ever generating $\square$, in this case the input set of clauses in satisfiable.
3. Saturation will run forever, but without generating $\square$. In this case the input set of clauses is satisfiable.

Saturation Algorithm in Practice

In practice there are three possible scenarios:

1. At some moment the empty clause $\square$ is generated, in this case the input set of clauses is unsatisfiable.
2. Saturation will terminate without ever generating $\square$, in this case the input set of clauses in satisfiable.
3. Saturation will run until we run out of resources, but without generating $\square$. In this case it is unknown whether the input set is unsatisfiable.

Inference Selection by Clause Selection



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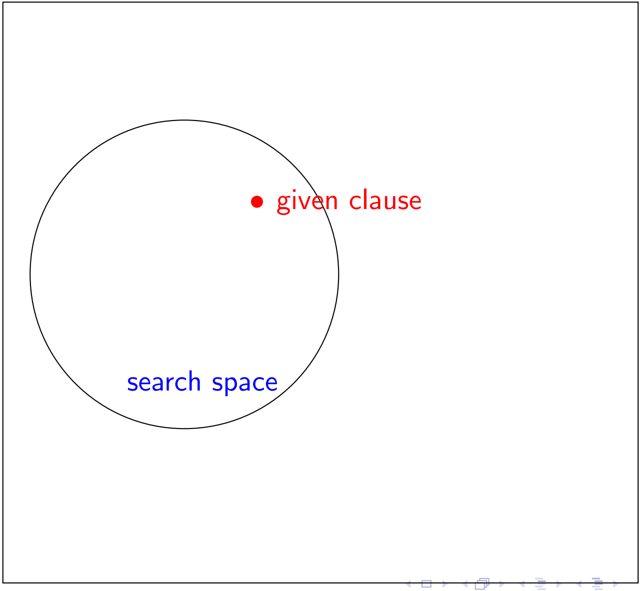

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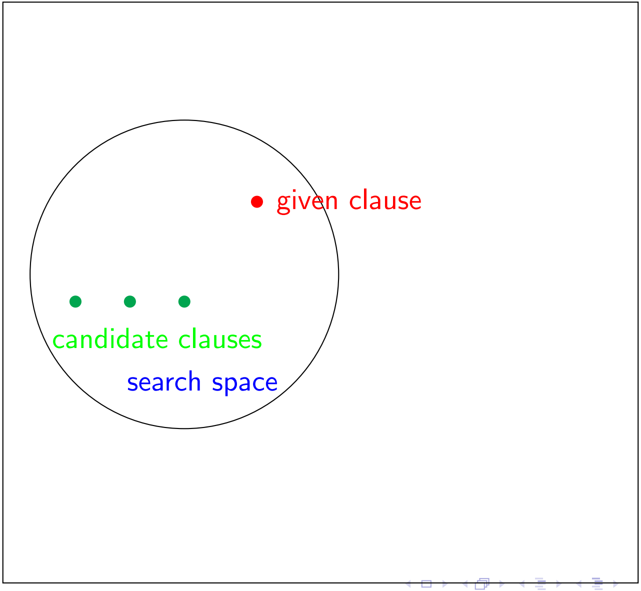

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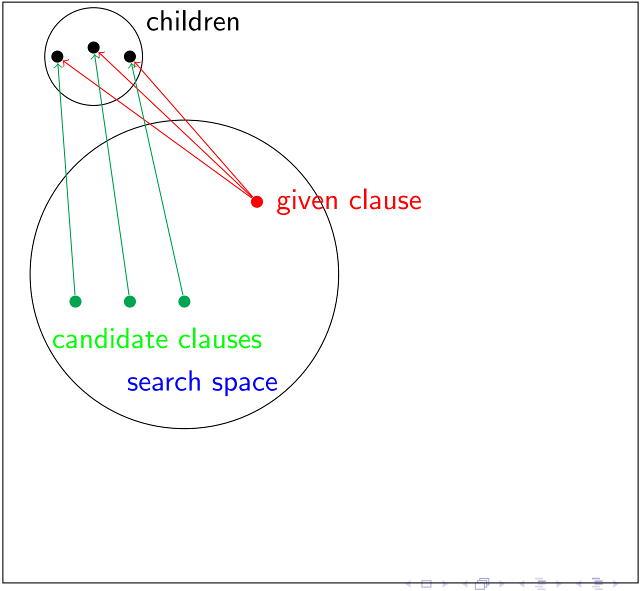

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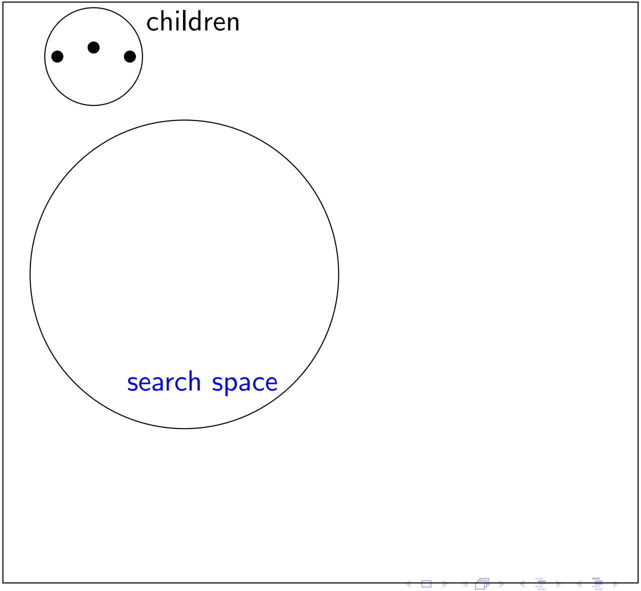

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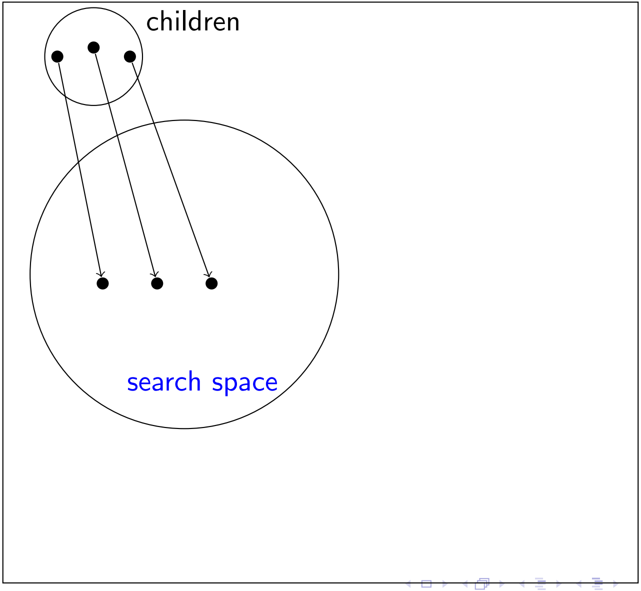

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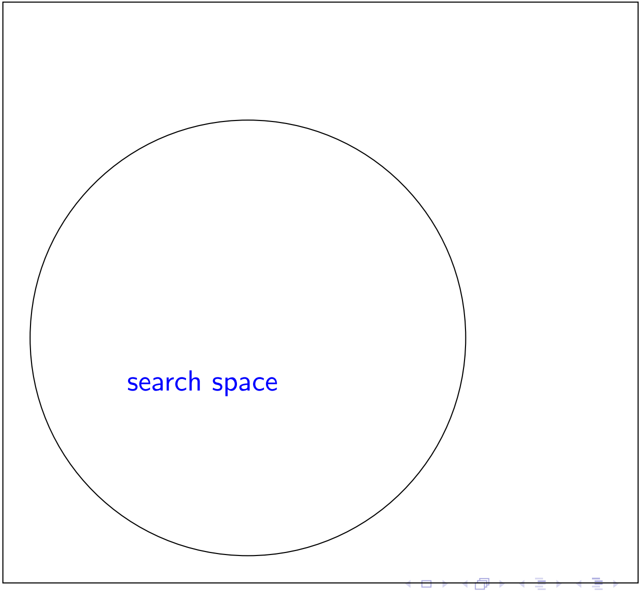

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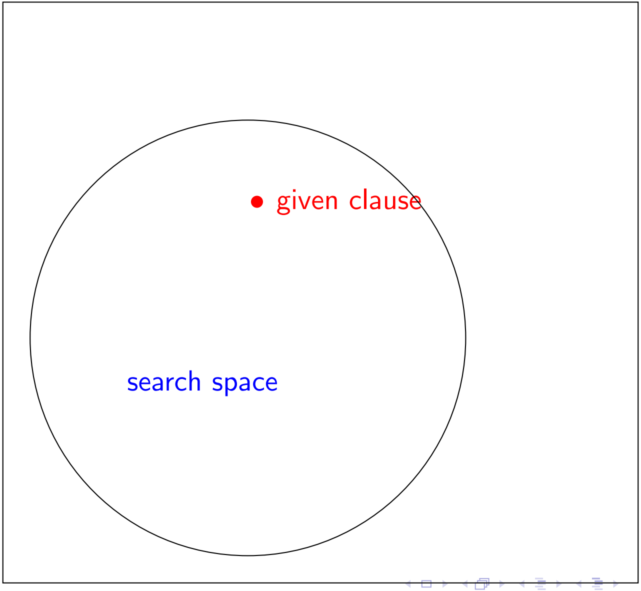

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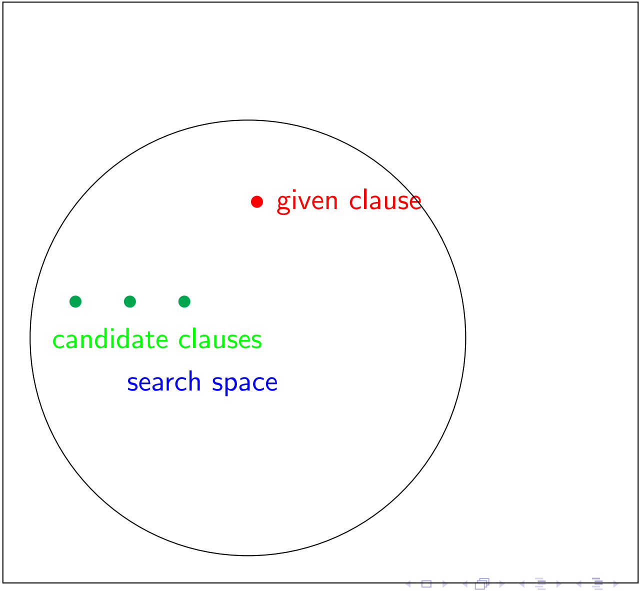

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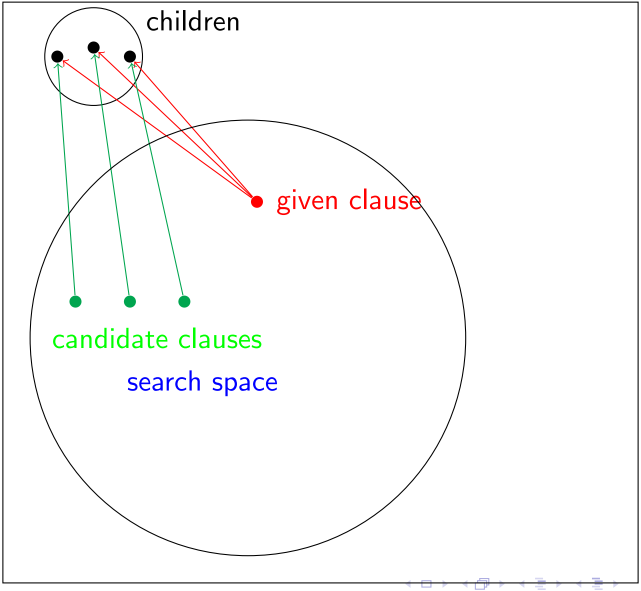

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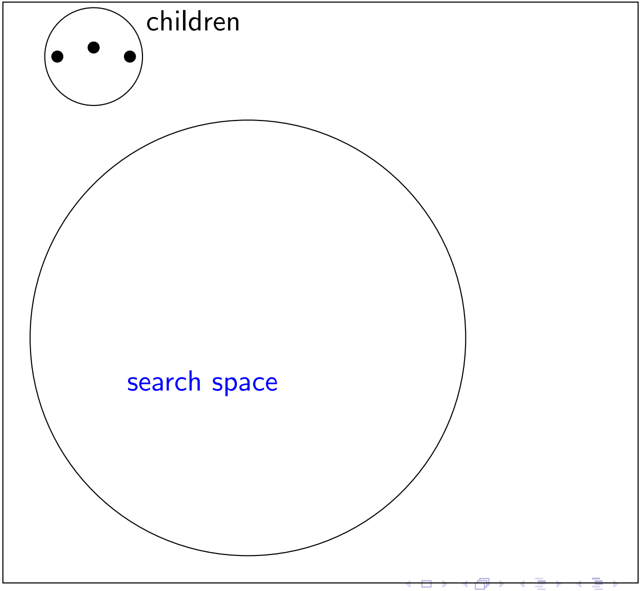

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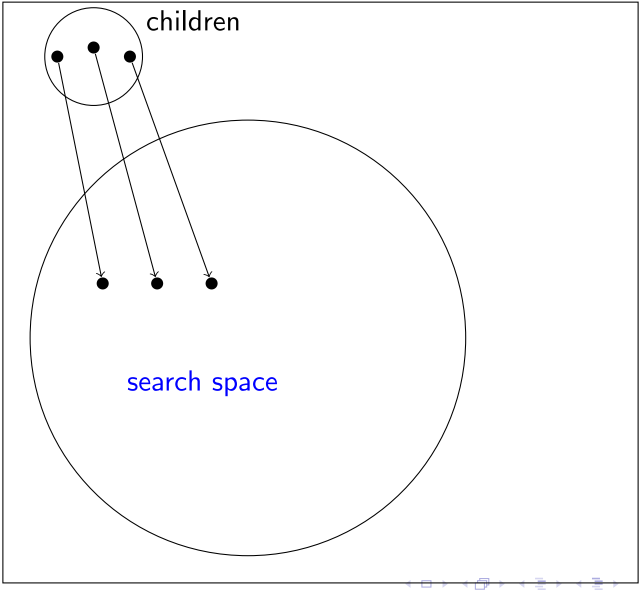

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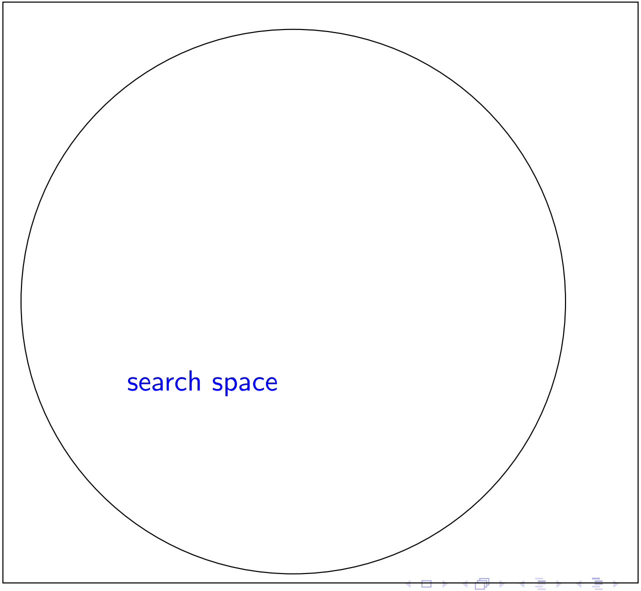

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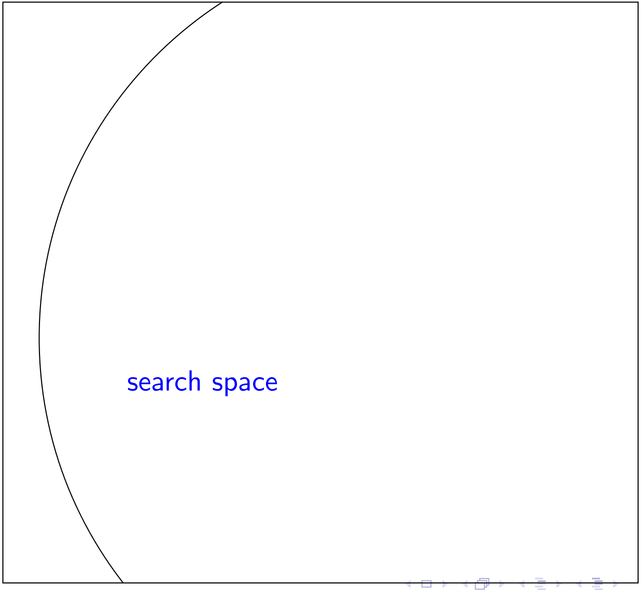

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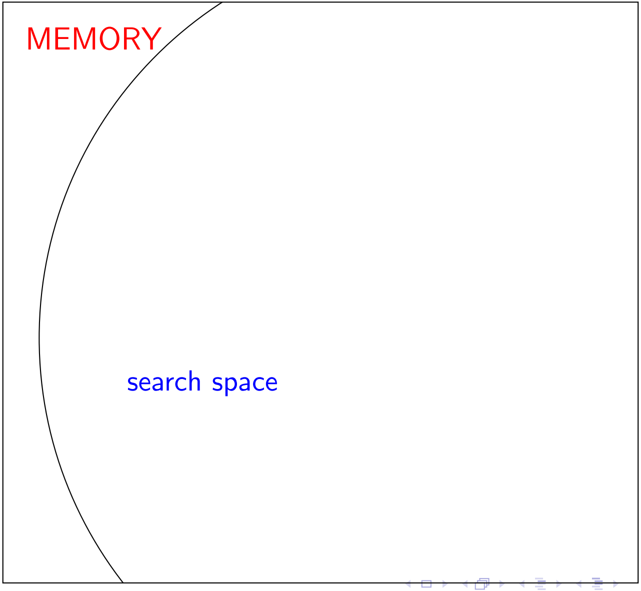

Saturation with the Given-Clause Algorithm

Only apply inferences to the *selected clause and the previously selected clauses*.

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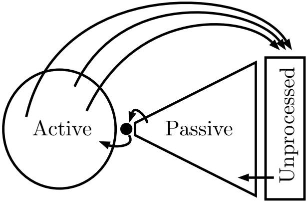

Thus, the search space is divided in two parts:

- active clauses, that participate in inferences;
- passive clauses, that do not participate in inferences.

Observation: the set of passive clauses is usually considerably larger than the set of active clauses, often by 2-4 orders of magnitude (depending on the saturation algorithm and the problem).

Anatomy of Saturation

Active

- initially empty
- backed up by sophisticated data structures (indexes) to allow efficient processing of inferences

Passive

- initially contains the clausified input
- typically consists of several queues ordering clauses to process by various (heuristical) criteria
- fairness!

Unprocessed:

- a temporary container
- just after generation, simplify before put into passive

The Clause Selection Task

Selecting the given clause is arguably the most important choice point in the implementation of a saturation algorithm

- If we only knew which to select up front . . .
- the standard approach: two queues (age, weight) and a ratio
- a natural spot for applying ML

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- a natural spot for applying ML

Notable attempts so far:

- Deep Network Guided Proof Search. LPAR 2017
- ENIGMA: Efficient Learning-Based Inference Guiding Machine. CICM 2017
- much more work done since (Jan will tell)

Outline

Making It Fast in Practice

- Literal selection and ordering constraints
  - restrict applicability of inference rules
- Redundancy elimination and simplifications
  - tautology deletions, subsumption, demodulation
- Saturation loop variants
  - Otter loop, Discount loop, LRS
- The AVATAR architecture
- Efficient data structures: term sharing, indexing, ...
- Specialised modes and calculi: InstGen, FMB, . . .
- . . .
- Strategy scheduling mode

Options and Strategies

A typical theorem prover has many ways to set up and run the proving process.

A naive idea: leave it up to the user to pick the best option setup, i.e. a strategy, for the problem P at hand.

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A more fruitful idea:
Automatically run a full schedule of strategies, ideally selected to have complementary strengths/weaknesses such that they cover the most problems.

- Introduced in Gandalf, (Tammet 1998)
- Vampire's famous *CASC mode*

Two more machine learning tasks:

Automatic Strategy Selection

- Given a problem P pick a strategy most likely to succeed on P
- e.g. MaLeS: A Framework for Automatic Tuning of Automated Theorem Provers. J. Autom. Reasoning 2015

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Automatic Strategy Invention

- Automatically discover sets of (complementary) strategies that together solve many problems (over a given benchmark)
- BliStr: The Blind Strategymaker. GCAI 2015
- BliStrTune: hierarchical invention of theorem proving strategies. CPP 2017

Outline

Thank you!

Questions?