Prover9 and Its Application to Challenging Problems in Mathematics

Symbolic Guidance with Proof Sketches and Hints

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April 2020

Overview

- Resolution-style theorem proving
- Look and feel (Prover9)
- Searching for a proof
- Advanced methods and features
- Applications

Resolution-style Theorem Proving

- First-order logic with equality
- Problem representation
  - language of clauses
  - proof by contradiction
- Inference rules
  - resolution (modus ponens / syllogism)
  - paramodulation (equality substitution)
  - unification
- Demodulation (rewriting)
- Subsumption (deletion)

Logical equivalences

$$p \to q \iff \neg p \vee q$$

$$(p \wedge q) \to r \iff \neg p \vee \neg q \vee r$$

Implicit quantification and scope of variables

```
-LT(x,y) | -LT(y,z) | LT(x,z).
-LT(x,y) | -LT(y,x).
```

$$(\forall x \forall y \forall z(\ \neg LT(x,y) \vee \neg LT(y,z) \vee LT(x,z)\ )$$

$$\wedge$$

$$(\forall x \forall y(\ \neg LT(x,y) \vee \neg LT(y,x)\ )$$

Inference Rules

Resolution

```
   -P(x,b) | Q(x).   {a/x}
   P(a,x) | R(x).    {b/x}
  ----------------
   Q(a) | R(b).
```

Hyperresolution

```
   -P(x,y) | -Q(x,y) | R(x,y).    {a/x,b/y}
   P(a,x).                        {b/x}
   Q(x,b).                        {a/x}
  -----------------------------
   R(a,b).
```

Paramodulation

```
   P(f( a * x, g(x)))   {b/x}
        x * b = x       {a/x}
   ------------------
   P(f( a    , g(b)))
```

Demodulation and Subsumption

- Demodulation: simplify and canonicalize
  - replace all instances of term x + 0 with the corresponding instance of term x
  - right associate all expressions
- Subsumption: discard less general information

  $0 + 2 = 2, 0 + 3 = 3, ...$

  are subsumed by

  $0 + x = x$

Identification of demodulators depends on an underlying ordering of terms.

The Task

Given an initial set C of clauses and a set of inference rules, find a derivation of the *empty clause* (for example, by the resolution of two conflicting clauses `P` and `-P`).

Procedure:

```
while (no proof found)
{
  select "given" clause G
  apply inference rules to G together with
    clauses from {have been given}
  process inferred clauses (demodulation, subsumption)
}
```

Some provers delay processing inferred clauses until chosen as given.

Look and Feel

Example input files ...

ATP Research Objectives

For resolution-style provers ...

- Automatic theorem proving
  - fully automated
  - consistently and reliably prove "easy" problems easily
- Prover as a research tool
  - part of a process
  - mathematically challenging problems (e.g., open questions)

Successful vs. Failed Searches

Choice of representation, inference rules (e.g., which variations of resolution to use), rewriting and deletion strategies all matter, but it mostly comes down to *given selection*.

Given selection is the focus of most research activity (e.g., the development of machine learning methods).

Given Selection

Methods

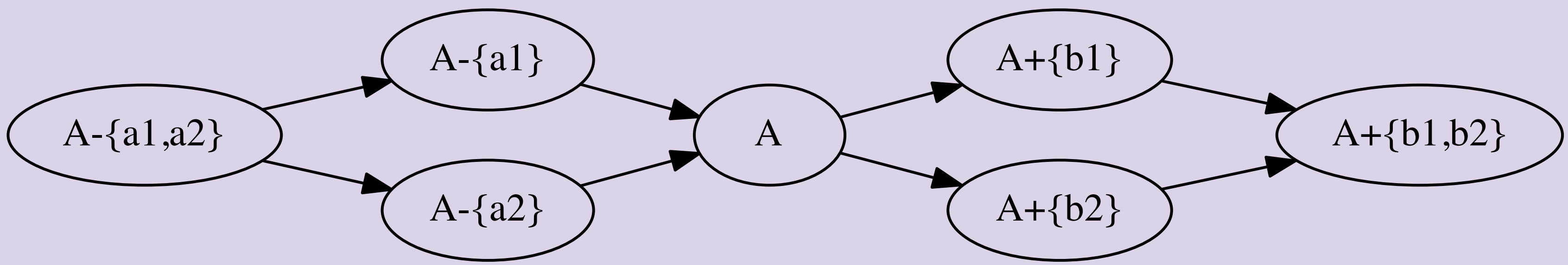
- Symbol count (weighting)
- User-defined weighting patterns
- Attribute-based selection
- Subsumption-based selection (hints)
- Model-based selection (semantic guidance)
- Statistical methods (e.g., machine learning)

The user can specify detailed recipes for combining these mechanisms, including rules based on clause properties.

Applications to Math Research

- Collaboration with mathematicians to help them solve their research problems
- Working toward a key result often requires (human) planning, multiple runs of the prover and several intermediate results
- Outcomes
  - new math results, solutions to open questions
  - new features supported by the theorem provers
  - new analysis and support tools
  - improved expertise and methods for using the tools in the most effective way

Say we want to prove a theorem t in a target theory A.



We can learn given-selection strategies by looking at

- proofs of t in extensions of A (*proof sketches*)
- countermodels of t in weakenings of A (*semantic guidance*)

Proof Sketches

Consider a derivation of some c_n as a sequence of clauses,

$$c_1, c_2, \ldots, c_i, \ldots, c_j, \ldots, c_n$$

where

- c_i is an extra assumption not in the target theory A
- derived clause c_j has c_i in its derivation history

c_j either is derivable from A or it is not.

- if yes, it suffices to find a new derivation of c_j
- if no, it suffices to "bridge the gaps" to the consequences of c_j

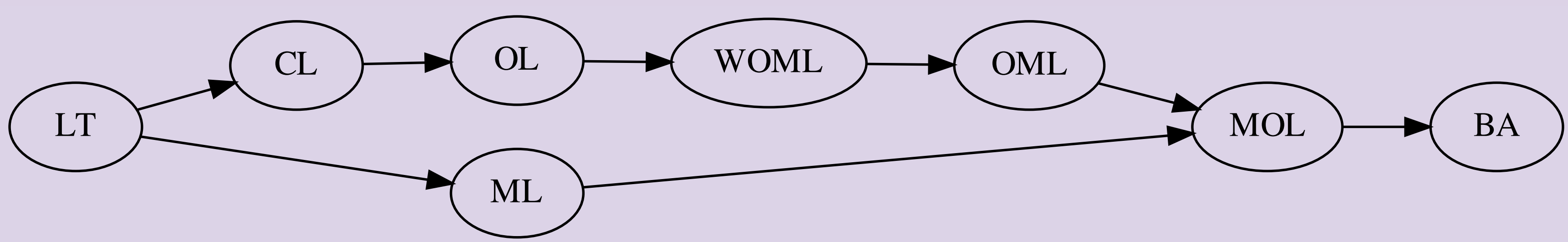
In either case, we have a partial proof that *might* be easier to complete than finding a proof from scratch.

The Proof Sketches Method

- Idea: Collect proofs of the target theorem in extended theories (i.e., with extra assumptions) and have a selection bias for clauses that match clauses in these proofs.
- The emphasis is on the *sufficiency* of the collected "proof sketches". This does not preclude finding a different proof.
- Move up the hierarchy by systematically generating new proof sketches with fewer extra assumptions, including all previous proof sketches for guidance.
- The challenge is to find effective extensions of the target theory (extra assumptions).

Where Do Extra Assumptions Come From?

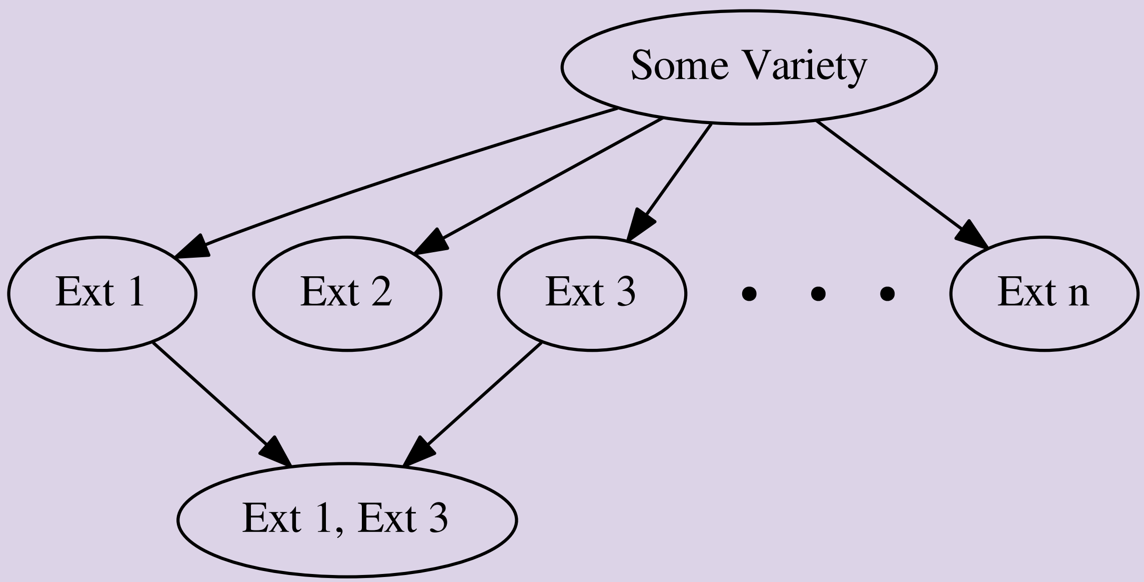
Example: Lattice Theory Hierarchy



$\mathcal{LT}$

- + Invertibility ($\mathcal{CL}$)
- + Compatibility ($\mathcal{OL}$)
- + Weak Orthomodularity ($\mathcal{WOML}$)
- + Orthomodularity ($\mathcal{OML}$)
- + Modularity ($\mathcal{MOL}$)
- + Distributivity ($\mathcal{BA}$)

Other Extensions



Examples:

- $x * y = y * x$
- $(x * y) * z = x * (y * z)$
- $x * x = x$

Proof Sketches in Prover9

- Proof sketches can be included as *hints*.
- Given selection can be biased toward clauses that *match* (subsume) hints.
- Hints also can come from
  - the mathematician
  - proofs of related theorems in the same theory

Proving target theorems with multiple extra assumptions and then iteratively eliminating them has been an especially effective method for proving difficult theorems.

Semantic Guidance

- Say $A \Rightarrow c$ is a theorem but $A - \{a\} \Rightarrow c$ is *not* a theorem.
- Let I be an interpretation (model) that satisfies $A - \{a\}$ and falsifies c.
- Key observation: In order to infer c at least one parent p of the inference must evaluate to **False** under I. Similarly for the parents of p, and so on ...
- It follows that a proof of c from A will necessarily include steps that evaluate to **False** under I.

  ... a and a subset of a's descendants.

- Idea: Have some selection bias for clauses that evaluate to **False** under I.
- The challenge is to find weakenings of A that yield good candidate interpretations I.

  ... want a to be minimal.

Semantic Guidance in Prover9

- Mace4 can be used to find finite models and counterexamples.
- Prover9 can include the resulting interpretations as input.
- The user can specify how to use the evaluation of clauses for given selection.